SOME RELATIONS CONNECTING THE SUMS OF THE COAXIAL MINORS OF A CIRCULANT.

By W. H. Metzler.

In 1878 Dr. J. W. S. Glashier called attention* to the fact that the determinant of order n formed from subtracting x from the elements along the principal diagonal of a continuant has as factors:

$$-(x-s)\prod_{k=1}^{m}\{x^2-(a_1+a_2\theta^k+a_3\theta^{2k}+\cdots+a_n\theta^{(n-1)k}) \times (a_1+a_2\theta^{n-k}+a_3\theta^{n-2k}+\cdots+a_n\theta^{n-(n-1)k})\}$$

for

$$n=2m+1,$$

and

$$(x-s)(x-s')\prod_{k=1}^{m-2}\{x^2-(a_1+a_2\theta^k+a_3\theta^{2k}+\cdots+a_n\theta^{(n-1)k}) \times (a_1+a_2\theta^{n-k}+a_3\theta^{n-2k}+\cdots+a_n\theta^{n-(n-1)k})\}$$

for

$$n=2m,$$

where

$$s=a_1+a_2+a_3+\cdots$$
$$s'=a_1-a_2+a_3-\cdots$$

and θ is an imaginary nth root of unity.

Thus for $n=5$ we have

$$\begin{vmatrix} a_1-x & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3-x & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5-x & a_1 & a_2 \\ a_4 & a_5 & a_1 & a_2-x & a_3 \\ a_5 & a_1 & a_2 & a_3 & a_4-x \end{vmatrix}$$

$$=-\{x-(a_1+a_2+a_3+a_4+a_5)\}\{x^2-(a_1+a_2\theta+a_3\theta^2 + a_4\theta^3+a_5\theta^4)(a_1+a_2\theta^4+a_3\theta^3+a_4\theta^2+a_5\theta)\} \times\{x^2-(a_1+a_2\theta^2+a_3\theta^4+a_4\theta+a_5\theta^3)(a_1+a_2\theta^3 + a_3\theta+a_4\theta^4+a_5\theta^2)\}.$$

* *Quarterly Journal of Mathematics,* Vol. XV, pp. 347–356. Cf. Muir, *Messenger of Mathematics,* New Series, No. 491, 1912.

If we represent the imaginary factor of the circulant by α_1, α_2, α_3, α_4, the right-hand side may be written

$(x-s)(x^2-\alpha_1\cdot\alpha_4)(x^2-\alpha_2\cdot\alpha_3)$ or $(x-s)(x^2-\alpha_{1,4})(x^2-\alpha_{2,3})$,
where

$$\alpha_{i,j}=\alpha_i\cdot\alpha_j.$$

Expanding both sides in terms of powers of x, we have in the general case
for $$n=2m,$$

$$(A)\quad \begin{aligned} &x^{2m}-x^{2m-1}\Sigma a_{11}+x^{2m-2}\Sigma\binom{12}{12}-\cdots-x\cdot\Sigma A_{11}+\Delta \\ &\quad = x^{2m}-x^{2m-1}(s+s')+x^{2m-2}(s\cdot s'-\Sigma\alpha_{1,n-1})+\cdots \\ &\qquad +(-1)^{k-1}x^{2m-2k}(s\cdot s'\Sigma\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{k-1,n-k+1} \\ &\qquad -\Sigma\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{k,n-k}) \\ &\qquad +(-1)^{k-1}x^{2m-2k-1}(s+s')\Sigma\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{k,n-k}+\cdots \\ &\qquad +(-1)^{m-1}s\cdot s'\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{m-1,n-m+1};\end{aligned}$$

for $$n=2m+1,$$

$$(B)\quad \begin{aligned} &-x^n+x^{n-1}\Sigma a_{11}-x^{n-2}\Sigma\binom{12}{12}+\cdots x\cdot\Sigma A_{11}+\Delta \\ &\quad = -x^n+x^{n-1}s+x^{n-2}\Sigma\alpha_{1,n}-\cdots \\ &\qquad +(-1)^{k-1}x^{n-2k}\cdot\Sigma\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{k,n-k} \\ &\qquad +(-1)^{n-1}x^{n-2k-1}\cdot s\cdot\Sigma\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{k,n-k}+\cdots \\ &\qquad\qquad +(-1)^m\cdot s\cdot\alpha_{1,n-1}\cdot\alpha_{2,n-2}\cdots\alpha_{m,n-m};\end{aligned}$$

Where Σa_{11} represents the sum of the elements along the principal diagonal of the circulant Δ, $\Sigma\binom{12\cdots k}{12\cdots k}$, the sum of the coaxial minors of order k, and ΣA_{11} the sum of the coaxial minors of order $n-1$.

Equating coefficients of like powers of x we have from (A)

$$(s+s')\Sigma\alpha_{1,n-1}\cdots\alpha_{k-1,n-k+1}=(-1)^{k-1}\Sigma\binom{12\ldots 2k-1}{12\ldots 2k-1},$$

$$s\cdot s'\Sigma\alpha_{1,n-1}\cdots\alpha_{k-1,n-k+1}-\Sigma\alpha_{1,n-1}\cdots\alpha_{k,n-k}=(-1)^{k-1}\Sigma\binom{12\ldots 2k}{12\ldots 2k},$$

$$(s+s')\Sigma\alpha_{1,n-1}\cdots\alpha_{k,n-k}=(-1)^k\Sigma\binom{12\ldots 2k+1}{12\ldots 2k+1},$$

that is,

$$(-1)^{k-1}s\cdot s'\frac{\Sigma\binom{12\ldots 2k-1}{12\ldots 2k-1}}{s+s'}-(-1)^k\frac{\Sigma\binom{12\ldots 2k+1}{12\ldots 2k+1}}{s+s'}=(-1)^{k-1}\Sigma\binom{12\ldots 2k}{12\ldots 2k}.$$

or

$$s \cdot s' = \frac{(s + s')\Sigma\binom{12 \ldots 2k}{12 \ldots 2k} - \Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}}{\Sigma\binom{12 \ldots 2k-1}{12 \ldots 2k-1}}, \tag{1}$$

a constant ratio for all values of k from 1 to $m - 1$.

When $k = m$, the ratio becomes

$$\frac{\Sigma a_{11} \cdot \Delta}{\Sigma A_{11}};$$

from B,

$$\Sigma \alpha_{1,\, n-1} \cdots \alpha_{k,\, n-k} = (-1)^k \Sigma\binom{12 \ldots 2k}{12 \ldots 2k},$$

$$s \cdot \Sigma \alpha_{1,\, n-1} \cdots \alpha_{k,\, n-k} - (-1)^k \Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1},$$

That is

$$s \cdot \Sigma\binom{12 \ldots 2k}{12 \ldots 2k} = \Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}$$

or

$$s = \frac{\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}}{\Sigma\binom{12 \ldots 2k}{12 \ldots 2k}}. \tag{2}$$

In the case where $n = 2m$, we have

$$s' = \frac{\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}}{\Sigma\binom{12 \ldots 2k}{12 \ldots 2k}}, \quad \text{if} \quad s = 0, \tag{1a}$$

It is known* that when $s = 0$ all the primary minors of Δ are equal. It follows, therefore, that the unique primary minor

$$A_1 = \frac{s' \cdot \Sigma\binom{12 \ldots 2m-2}{12 \ldots 2m-2}}{2m}. \tag{1b}$$

Similarly

$$s = \frac{\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}}{\Sigma\binom{12 \ldots 2k}{12 \ldots 2k}} \quad \text{when} \quad s' = 0. \tag{1c}$$

If $s + s' = 0$, then

$$s^2 = s'^2 = \frac{\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1}}{\Sigma\binom{12 \ldots 2k-1}{12 \ldots 2k-1}}. \tag{1d}$$

* Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Crelle's Journal,* lvii, pp. 111–121. Cf. Muir, *Messenger of Mathematics,* New Series, No. 536, Vol. XLV, December, 1915.

If $s = s' = 0$, then

$$\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1} = 0. \tag{1e}$$

From $(1b)$ we see that, when $s = s' = 0$, the unique minor A_1 vanishes. In this case the determinantal equation contains only even powers of x and since all its roots are real it follows that the signs must be alternately positive and negative. That is, the signs of $\Sigma\binom{12}{12}$, $\Sigma\binom{1234}{1224} \cdots$ must be alternately negative and positive.

In the case of $(1d)$ we see that the sums of the coaxial minors of odd order have the same sign.

In the case when $n = 2m + 1$ we have

$$\Sigma\binom{12 \ldots 2k+1}{12 \ldots 2k+1} = 0 \quad \text{of} \quad s = 0. \tag{2a}$$

Here again the signs of the sums of coaxial minors of even order must be alternately negative and positive.

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